

$$\underline{S^2 \times S^2}$$

Background

Differential Geometry notation:

If $(u, v, \dots)$ are coordinates on a surface (or manifold), then e.g. $\partial_u = \frac{\partial}{\partial u}$ = velocity vector field of the coordinate u . If $f: M \rightarrow \mathbb{R}$ is a fcn on your surface/manifold, $\partial_u f = \frac{\partial f}{\partial u}$ = directional derivative of f in direction u .

Example: Spherical coordinates on S^2

coordinates chart: $F(\theta, \varphi) = (\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi)$
 $(\theta, \varphi) \in (0, \pi), (0, \pi)$

$$\partial_\theta = (-\sin \theta \sin \varphi, \cos \theta \sin \varphi, 0)$$

$$\partial_\varphi = (\cos \theta \cos \varphi, \sin \theta \cos \varphi, -\sin \varphi)$$

Gauss curvature $K = K_1 K_2$
of surface



Using Riemann curvature tensor

$$K = \frac{\langle R(u, v)v, u \rangle}{\langle u, u \rangle \langle v, v \rangle - \langle u, v \rangle^2}$$

where u, v are two tangent vectors that are linearly independent,

Here, $\langle u, v \rangle = \text{metric}$

$$R(u, v) = \nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u, v]}$$

↑ covariant derivative

$$\nabla_u^{(\cdot)} = \Pi_{\text{Target}} U^{(\cdot)}$$

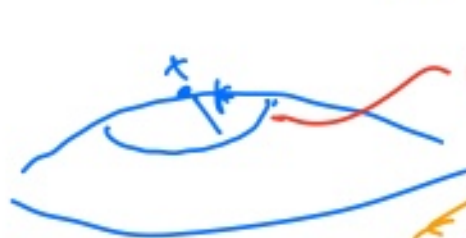
Can be computed completely in terms of the metric.

If v, w, x are v. fields

$$V \langle w, x \rangle = \langle \nabla_v w, x \rangle + \langle w, \nabla_v x \rangle$$

Another way to think of Gauss curvature $K(x)$

↑
 $x \in M.$



$$\begin{aligned} \text{circumference} \\ = 2\pi r \left(1 - \frac{K r^2}{6} + O(r^3) \right) \end{aligned}$$

on S^2
unit sphere
circumf = $2\pi \sin(r)$

Sectional Curvature -
on an n -dim manifold M at $p \in M$

$$K_p(V, W) = \frac{\langle R(V, W)W, V \rangle}{\langle V, V \rangle \langle W, W \rangle - \langle V, W \rangle^2}$$

where $R(V, W) = \nabla_V \nabla_W - \nabla_W \nabla_V - \nabla_{[W, V]}$

where V, W are lin. indep. vectors in $T_p M$.

$K_p(V, W) \equiv$ Gauss curvature of the surface spanned
by V, W ,

(really take plane spanned by V, W ,
look at its image on the manifold
via the exponential.)

Riem.

A manifold (M, g) is said to have

positive sectional curvature (PSC) if $\forall p \in M$,

$\forall$ tangent 2-planes at p , $K_p(X, Y) > 0$.

(and we call g a PSC metric).

Conjecture of Hopf. There does not exist
a PSC metric on $S^2 \times S^2$.

Standard metric on $S^2 \times S^2$.

metric on S^2 : (θ, ϕ)

$$g_{ij} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

$$g_{ij} = \langle \partial_i, \partial_j \rangle$$

$$\partial_1 = \partial_\theta, \partial_2 = \partial_\phi,$$

$$\Rightarrow g = \begin{pmatrix} \sin^2 \phi & 0 \\ 0 & 1 \end{pmatrix}$$

on $S^2 \times S^2$ $(\theta_1, \phi_1, \theta_2, \phi_2)$

$$g = \begin{pmatrix} \sin^2 \phi_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sin^2 \phi_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Covariant derivatives on S^2

$$\nabla_{\partial_\phi} \partial_\phi = 0, \quad \nabla_{\partial_\phi} \partial_\theta = \nabla_{\partial_\theta} \partial_\phi = \cot \phi \partial_\theta$$

$$\nabla_{\partial_\theta} \partial_\theta = -\sin \phi \cos \phi \partial_\phi$$

Covariant derivatives on $S^2 \times S^2$

$$\nabla_{\partial_{\theta_i}} \partial_{\theta_i} = -\sin \phi_i \cos \phi_i \partial_{\phi_i}$$

$$\sin \cdot \nabla_{\partial_{\phi_i}} \partial_{\theta_i} = \dots$$

$$\nabla_{\partial_{\phi_i}} \partial_{\phi_i} = 0$$

$$\nabla_{\partial_{\theta_1} \text{ or } \partial_{\phi_1}} \partial_{\theta_2 \text{ or } \partial_{\phi_2}} = 0$$

Sectional curvature of $S^2 \times S^2$

$$K_p(\partial_{\theta_1}, \partial_{\phi_1}) = 1$$

$$K_p(\partial_{\theta_2}, \partial_{\phi_2}) = 1$$

$$K_p(\text{any}_1, \text{any}_2) = 0$$

The standard metric on $S^2 \times S^2$ is not $PS_2 C_n$.

Bourguignon — showed that if we consider a family $g_t = \underset{\substack{\text{standard} \\ S^2 \times S^2}}{g_0} + t \underset{\substack{\text{symm. 2-tensor}}}{h} + O(t^2)$

Then $\frac{\partial}{\partial t} K_{\underset{\substack{\uparrow \uparrow \\ \text{mixed}}}}(X, Y) = 0$

Berger's 1st order obstruction

$$M_{g_0} = M_1 \times M_2 \text{ product metric}$$

$$g_t = g_0 + t h + O(t^2)$$

$$X \in TM_1, Y \in TM_2,$$

$$\begin{aligned} K'(X, Y) &= (\nabla_X \nabla_Y h)(X, Y) - \\ \downarrow = \frac{\partial}{\partial t} & - \frac{1}{2} (\nabla_X \nabla_X h)(Y, Y) \\ & - \frac{1}{2} (\nabla_Y \nabla_Y h)(X, X). \end{aligned}$$

Bourgaignon proved:

$$\text{Fix } X \in T_p M_1, Y \in T_p M_2$$

unit vectors that are orthogonal w.r.t g_0

let $\gamma: S^1 \rightarrow M_1, \eta: S^1 \rightarrow M_2$ be
two closed geodesics s.t. $\gamma'(0) = X, \eta'(0) = Y$

$$\nabla_X X = \nabla_Y Y = 0 \quad [X, Y] = 0$$

Use Berger's formula

$$K'(x, y) = 2_s \partial_t h(x, y) - \frac{1}{2} \partial_s^2 h(y, y) - \frac{1}{2} \partial_t^2 h(x, x)$$

$$\Rightarrow \int_{S' \times S'} K'(x, y) ds dt = 0.$$

If $K'(x, y) \geq 0$ $\forall$ x, y in mixed plane in $T_p M$,
then $K'(x, y) = 0$.

What about higher order terms, eg

$$K_{g_t}(x, y) = t^2 (A(x, y)) + O(t^3)$$

(BDS) Bourguignon - Deschamps - Senenec 1972-73

Let (M, g_0) be a cpt Riemannian product manifold

$$M = M_1 \times M_2$$

$$g_0 = g_1 + g_2$$

with no nonzero Killing fields - *no where zero* $\nearrow$ *vector field coming from the derivative of a one-parameter family of isometries.*

Then there does not exist an analytic one-parameter family of metrics $g_t = g_0 + t h_1 + t^2 h_2 + \dots$

Such that $K_{g_t} > 0$ $\forall$ sufficiently small $t > 0$.

Idea of Proof of BDS

Sp. $X \in TM_1, Y \in TM_2$

$$(M, g_0) = (M_1, g_1) \times (M_2, g_2)$$

$$\text{Let } \pi = \text{span} \{X, Y\} \Rightarrow K_{g_0}(X, Y) = 0.$$

Consider a C^r variation of metrics

$$g(t) = g_0 + t h^1 + \frac{t^2}{2} h^2 + \dots + \frac{t^r}{r!} h^r + o(t^r).$$

$$\text{i.e. } h^j = \left. \frac{d^j g}{dt^j} \right|_{t=0}$$

Issue: If ϕ_t is a family of diffeos $\phi_t: M \rightarrow M$,
then $\phi_t^* g_0 \rightarrow$ will have exactly the same sectional
curvatures (gauge invariance).

Defn BDS calls a variation of metrics
"r-positive" if the first nonzero derivative ^{of order} $K_{g_t}(X, Y)$ is
 > 0 . A variation is called "r-product" if every
coefficient $h^j = p_1^* a_j + p_2^* b_j$ where $\forall j \leq r$
 $a_j \in \Gamma(S^2 T^* M_1) \leftarrow \in \Gamma(S^2 T^* M_2)$

Thm Sp. $M_1 \times M_2$ has no infinitesimal isometries.

Thm, every r-positive deformation of the product metric
is r-product after tweaking the gauge w/ diffeos

Thm - Sp. $M \times M$ has no infinitesimal isometries
 Thm $\nexists$ an analytic vector $g(t)$ st. $\langle g, g \rangle > 0$ $\forall$ small $t > 0$.

Idea of Proof:

Let S be the operator

$$Sh(X, Y) = \frac{d}{dt} \bigg|_{t=0} K_{g+th}(X, Y)$$
 where X, Y spans a mixed plane.

(up to normalization so that $|X \wedge Y|^2 = 1$).

Let $X = X^i \partial_i$, $Y = Y^\alpha \partial_\alpha$

By Berger formula

$$Sh(X, Y) = \left(\nabla_i \nabla_\alpha h_{j\beta} - \frac{1}{2} \nabla_i \nabla_j h_{\alpha\beta} - \frac{1}{2} \nabla_\alpha \nabla_\beta h_{ij} \right) X^i X^j Y^\alpha Y^\beta$$

↑
Sums over repeated indices.

Lemma $Sh(X, Y) \geq 0 \quad \forall X, Y \Rightarrow Sh(X, Y) = 0 \quad \forall X, Y$
 (Berger) mixed mixed

But $Sh = 0$ does not imply h is a product tensor.
 $h = L_V g_0$ if V is the vector field from a family of diffeom.

BDS used Berger-Ebin decomposition of $S^2 T^*M$

If $\delta : S^2 T^*M \rightarrow T^*M$ is the divergence
 i.e. $T \in S^2(T^*M)$

δT is the oneform defined by

$$\delta(T) = (\nabla_{e_i} T)(e_i, \cdot) \quad \text{where } \{e_i\} \text{ is an } n\text{-n frame.}$$

Let δ^* be the formal adjoint of δ

$$\delta^*: T^*M \rightarrow \text{Sym}^2 T^*M$$

If V^b is the one-form coming from an infinitesimal isometry V , then $\delta^*(V^b) = \frac{1}{2} \mathcal{L}_V g_0$

$\text{Im } \delta^* \subseteq \text{Sym}^2 T^*M$ is the set of h 's coming from infinitesimal isometries.

In the case where there are ~~non~~ nonzero Killing fields,

$$\text{Sym}^2 T^*M = \text{im } \delta^* \oplus \ker \delta$$

$$\text{Each of } h^i = \delta^* \xi^i + h^i_{\perp}$$

κ can force this to 0 by wiggling metric with pullbacks of diffeos along the way.